SISTEMI DINAMICI

[Fotocopie di Appunti]

A CURA DI ALESSANDRO PAGHI

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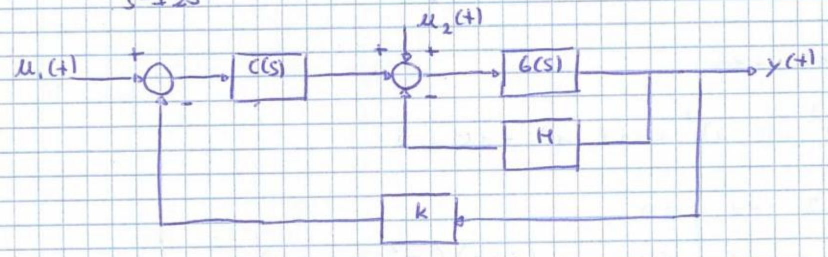
LINK AL CORSO ANNO 2014/2015:
http://www3.diism.unisi.it/FAC/index.php?bodyinc=didattica/inc.insegnamento.php&id=54893&aa=2014

FREQUENTAZIONE: Consigliata.

ES

$G(s) = \frac{1}{s^2+2s}$, $C(s) = \frac{10}{s}$, $K, H \in \mathbb{R}$

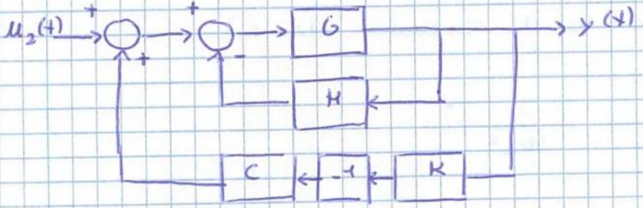


1) Funzione di trasferimento da $u_1(t)$ a $y(t)$.

$$P = \frac{G}{1+GH} \quad , \quad W_1 = \frac{CP}{1+CP\cdot K} = \frac{C\cdot\frac{G}{1+GH}}{1+C\cdot\frac{G}{1+GH}\cdot K} = \frac{CG}{1+GH+GCK}$$

$$= \frac{\frac{10}{s}\cdot\frac{1}{s^2+2s}}{1+\frac{1}{s^2+2s}\cdot H+\frac{10}{s}\cdot\frac{1}{s^2+2s}\cdot K} = \frac{10}{s^3+2s^2+H\cdot s+10K}$$

2) Funzione di trasferimento da $u_2(t)$ a $y(t)$.



$$P = \frac{G}{1+GH}$$

$$W_2 = \frac{P}{1-P(-CK)} = \frac{P}{1+PCK} = \frac{\frac{G}{1+GH}}{1+\frac{G}{1+GH}CK} = \frac{G}{1+GH+GCK}$$

$$= \frac{\frac{1}{s^2+2s}}{1+\frac{1}{s^2+2s}H+\frac{1}{s^2+2s}\cdot\frac{10}{s}\cdot K} = \frac{s}{s^3+2s^2+Hs+10K}$$

3) Per quali K e H, W_1 e W_2 sono stabili in senso BIBO.

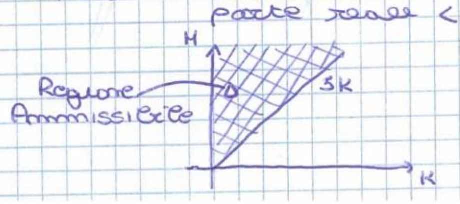
$Re[p_i] < 0$

Applico il criterio di Routh

3	1	H
2	2	$10K$
1	$\frac{2H-10K}{2}$	
0	$10K$	

$$\begin{cases} \frac{2H-10K}{2} > 0 \\ 10K > 0 \end{cases} \qquad \begin{cases} H > 5K \\ K > 0 \end{cases}$$

Sotto queste condizioni il polinomio ha soluzioni a parte reale < 0.



3) $K=1$, $H=11$, Diagrammi di Bode di $W_1(s)$ e $W_2(s)$.

$$W_1(s) = \frac{10}{s^3+2s^2+11s+10}$$

$$s=-1 \Rightarrow (-1)^3+2(-1)^2+11(-1)+10 = 2$$

$$\begin{array}{c|ccc|c} & 1 & 2 & 11 & 10 \\ -1 & & -1 & -1 & -10 \\ \hline & 1 & 1 & 10 & 0 \end{array}$$

$$(s+1)(\underbrace{s^2+s+10}_{\frac{-1\pm j\sqrt{39}}{2}})$$

In forma di Bode

$$W_1(s) = \frac{\cancel{10}}{(1+s)\cdot \cancel{10}\left(1+\frac{1}{10}s+\frac{1}{10}s^2\right)} = \frac{1}{(1+s)\left(1+\frac{1}{10}s+\frac{1}{10}s^2\right)}$$

$K_B = 1$

$g = 0$

Un polo reale $\tau = 1$ (punto di rottura $\omega = 1$)
a p.r. < 0

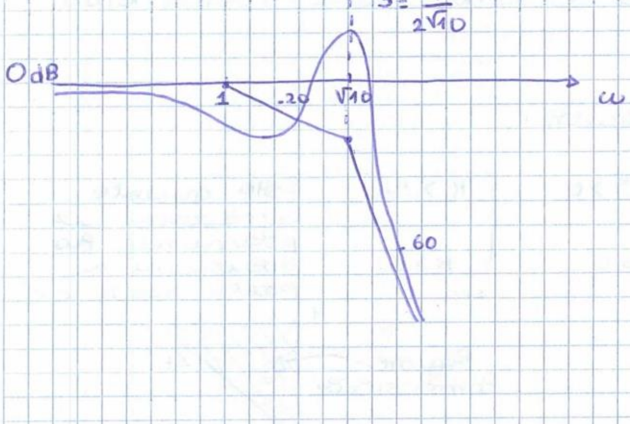
$$1+\frac{1}{10}s+\frac{1}{10}s^2 = 1+\frac{2\zeta}{\omega_n}s+\frac{1}{\omega_n^2}s^2$$

$$\omega_n = \sqrt{10}$$

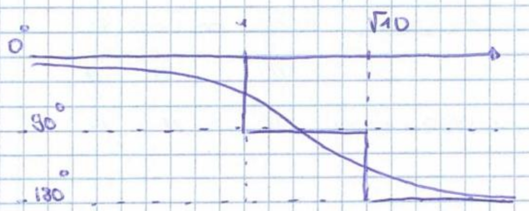
$$\frac{2\zeta}{\omega_n} = \frac{1}{10} \longrightarrow \zeta = \frac{1}{20}\omega_n = \frac{\sqrt{10}}{20} = \frac{1}{2\sqrt{10}}$$

e una coppia di poli complessi a parte reale negativa
(punti di rottura: $\omega_n = \sqrt{10}$, $\zeta = \frac{1}{2\sqrt{10}}$)

MODULO



FASE



4) $K = 1$, $H = 11$, determinare la risposta di regime permanente nell'uscita $y(t)$, relativa all'ingresso $u_2(t) = \cos(2t)$

$$W_2(s) = \frac{s}{s^3 + 2s^2 + 11s + 10}$$

$$u_2(t) = \cos(2t)$$

Il sistema è As. Stabile quindi $y_{PERM}(t)$ è definita.

$y_{PERM}(t)$

$$y_{PERM}(t) = |W_2(j2)| \cos(2t + \measuredangle W_2(j2))$$

$$W_2(j2) = \frac{j2}{(j2)^3 + 2(j2)^2 + 11(j2) + 10} = \frac{j2}{-8j - 8 + 22j + 10}$$

$$= \frac{j2}{2 + 14j} = \frac{j}{1 + 7j} = \frac{j(1 - 7j)}{50} = \frac{7 + j}{50}$$

$$|W_2(j2)| = \frac{1}{50}\sqrt{49 + 1} = \frac{1}{\sqrt{50}}$$

$$\measuredangle W_2(j2) = \arctan\left(\frac{1}{7}\right)$$

$$y_{PERM}(t) = \frac{1}{\sqrt{50}} \cos\left(2t + \arctan\left(\frac{1}{7}\right)\right)$$

ES

$x_1(k)$: frazione di elettori democratici dell'anno k.
$x_2(k)$: frazione di indecisi // // //.
$x_3(k)$: frazione di elettori repubblicani // // //.

$$x_1(k+1) = d\,x_1(k) + i_d\,x_2(k)$$
$$x_2(k+1) = (1-d)\,x_1(k) + (1-i_d-i_r)\,x_2(k) + (1-r)\,x_3(k)$$
$$x_3(k+1) = i_r\,x_2(k) + r\,x_3(k)$$

d : quota elettori democratici
r : quota elettori repubblicani
i_d : quota indecisi in k che diventano democratici in $k+1$
i_r : // // // // // // repubblicani // //

$d = r = 0,8$.

1) $i_d = i_r = 0,3$, Studiare la stabilità del sistema

$$A = \begin{pmatrix} d & i_d & 0 \\ 1-d & 1-i_d-i_r & 1-r \\ 0 & i_r & r \end{pmatrix} = \begin{pmatrix} 0,8 & 0,3 & 0 \\ 0,2 & 0,4 & 0,2 \\ 0 & 0,3 & 0,8 \end{pmatrix}$$

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda-0,8 & -0,3 & 0 \\ -0,2 & \lambda-0,4 & -0,2 \\ 0 & -0,3 & \lambda-0,8 \end{pmatrix}$$

$$= (\lambda-0,8)\left\{(\lambda-0,4)(\lambda-0,8) - 0,06\right\} + 0,2\left\{(-0,3)(\lambda-0,8)\right\} = 0$$

$$= (\lambda-0,8)(\lambda^2 - 0,4\lambda - 0,8\lambda + 0,32 - 0,06 - 0,06)$$

$$= (\lambda-0,8)(\lambda^2 - 1,2\lambda + 0,2)$$

$$= (\lambda-0,8)(\lambda-1)(\lambda-0,2) = 0.$$

autovalori del sistema:
$\lambda_1 = 1$, $\lambda_2 = 0,2$, $\lambda_2 = 0,8$

$\to$ Sistema marginalmente stabile.

Determinare la distribuzione asintotica dell'elettorato a partire da $X(0) = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$. Mostra che la dist. asintotica è sempre la stessa a prescindere da $X(0)$.

$X(0) = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} \qquad \lim_{k \to +\infty} X(k) = ?$

T. VALORE FINALE

$$\lim_{k \to +\infty} X(k) = \lim_{z \to 1} (z-1) X_e(z)$$

$$X_e(z) = z(zI - A)^{-1} X(0) = z \begin{pmatrix} z-0,8 & -0,3 & 0 \\ -0,2 & z-0,4 & -0,2 \\ 0 & -0,3 & z-0,8 \end{pmatrix}^{-1} \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

$$= \frac{z \cdot [\mathrm{Agg}(zI - A)]^T}{(z-1)(z-0,8)(z-0,2)} \cdot \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

$$= \frac{z \begin{pmatrix} z^2 - 1,2z + 0,32 - 0,06 & 0,3(z-0,8) & 0,06 \\ 0,2(z-0,8) & (z-0,8)^2 & 0,2(z-0,8) \\ 0,06 & 0,3(z-0,8) & z^2 - 1,2z + 0,32 - 0,06 \end{pmatrix}}{(z-1)(z-0,8)(z-0,2)} \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

$$\lim_{z \to 1} (z-1) X_e(z) = \frac{\begin{pmatrix} 0,06 & 0,06 & 0,06 \\ 0,04 & 0,04 & 0,04 \\ 0,06 & 0,06 & 0,06 \end{pmatrix}}{0,2 \cdot 0,8} \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

$$= \frac{1}{0,16} \begin{pmatrix} 0,06 \\ 0,04 \\ 0,06 \end{pmatrix} = \begin{pmatrix} 0,375 \\ 0,25 \\ 0,375 \end{pmatrix}$$

$$\begin{pmatrix} 0,06 & 0,06 & 0,06 \\ 0,04 & 0,04 & 0,04 \\ 0,06 & 0,06 & 0,06 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \qquad x, y, z = 1$$

$$= \begin{pmatrix} 0,06 \\ 0,04 \\ 0,06 \end{pmatrix} \frac{1}{0,16}.$$

ES. 3

$$\dot{x}_1(t) = \frac{x_1(t)\,x_2(t)}{x_2(t)+4} - u(t)$$

$$\dot{x}_2(t) = -\beta\,\frac{x_1(t)\,x_2(t)}{x_2(t)+4} + (1 - x_2(t))\,u(t)$$

$x_1(t)$: concentrazione batterica

$x_2(t)$: concentrazione substrato di reazione

$u(t)$: quantità di zucchero fornita

$u(t) = \frac{1}{2} \quad \forall t$

1) Determinare gli S.d.E. in funzione di $\beta \in \mathbb{R}$.

$$\begin{cases} \dfrac{x_1 x_2}{x_2+4} - \dfrac{1}{2} = 0 \\ -\beta\,\dfrac{x_1 x_2}{x_2+4} + (1 - x_2)\,\dfrac{1}{2} = 0 \end{cases}$$

$x_2 \neq -4$

$$\begin{cases} x_1 x_2 - \frac{1}{2}(x_2+4) = 0 \\ -\beta x_1 x_2 + \frac{1}{2}(1-x_2)(x_2+4) = 0 \end{cases}$$

$$\begin{cases} x_1 x_2 - \frac{1}{2}x_2 - 2 = 0 \\ -\beta x_1 x_2 + 2 - \frac{3}{2}x_2 - \frac{1}{2}x_2^2 = 0 \end{cases}$$

$$\begin{cases} x_1 = \dfrac{\frac{1}{2}x_2 + 2}{x_2} = \dfrac{1}{2} + \dfrac{2}{x_2}, \quad x_2 \neq 0 \\ -\beta\left(\frac{1}{2} + \frac{2}{x_2}\right)x_2 + 2 - \frac{3}{2}x_2 - \frac{1}{2}x_2^2 = 0 \end{cases}$$

$$-\frac{\beta}{2}x_2 - 2\beta + 2 - \frac{3}{2}x_2 - \frac{1}{2}x_2^2 = 0$$

$$\frac{1}{2}x_2^2 + \left(\frac{3}{2} + \frac{\beta}{2}\right)x_2 + 2\beta - 2 = 0$$

$$x_2^2 + (\beta+3)x_2 + 4(\beta-1) = 0$$

$$x_2 = \frac{-(\beta+3) \pm \sqrt{(\beta+3)^2 - 16(\beta-1)}}{2} \rightarrow \sqrt{\beta^2 + 6\beta + 9 - 16\beta + 16}$$

$$x_2 = \frac{-(\beta+3) \pm \sqrt{\beta^2 - 10\beta + 25}}{2} = \frac{-(\beta+3) \pm |\beta-5|}{2}$$

$$\beta > 5 \rightarrow x_2 = \frac{-(\beta+3) \pm (\beta-5)}{2} \begin{cases} -4 \quad \text{NO! } x_2 \neq -4! \\ -\beta + 1 \end{cases}$$

se $\beta < 5 \rightarrow x_2 = \dfrac{-(\beta+3) \pm (5-\beta)}{2} \Big\langle \begin{matrix} -\beta+1 \\ -4 \text{ NO!} \end{matrix}$

$$\begin{cases} x_2 = 1-\beta \\ x_1 = \dfrac{1}{2} + \dfrac{2}{x_2} = \dfrac{1}{2} + \dfrac{2}{1-\beta} = \dfrac{5-\beta}{2(1-\beta)} \end{cases}$$

$\exists!\ \bar{x} = \left(\dfrac{1}{2} + \dfrac{2}{1-\beta}, 1-\beta \right)$, con $\beta \neq 5, 1$

$\beta \neq 5$ perché $x_2 = 1-\beta \neq -4$.

2) Studiare la stabilità degli S.d.e. in funzione di $\beta \in \mathbb{R}$.

$$J = \frac{\partial f}{\partial x}\bigg|_{x=\bar{x}} = \begin{pmatrix} \dfrac{x_2}{x_2+4} & \dfrac{x_1(x_2+4)-x_1x_2}{(x_2+4)^2} \\ -\beta \dfrac{x_2}{x_2+4} & -\beta \dfrac{x_1(x_2+4)-x_1x_2}{(x_2+4)^2} - \dfrac{1}{2} \end{pmatrix}\Bigg|_{\begin{subarray}{l} x_1 = \frac{1}{2}+\frac{2}{1-\beta} \\ x_2 = 1-\beta \end{subarray}}$$

$$= \begin{pmatrix} \dfrac{x_2}{x_2+4} & \dfrac{4x_1}{(x_2+4)^2} \\ -\beta \cdot \dfrac{x_2}{x_2+4} & -\beta \cdot \dfrac{4x_1}{(x_2+4)^2} - \dfrac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1-\beta}{5-\beta} & \dfrac{2+\frac{8}{1-\beta}}{(5-\beta)^2} \\ -\beta \cdot \dfrac{1-\beta}{5-\beta} & -\beta \cdot \dfrac{2+\frac{8}{1-\beta}}{(5-\beta)^2} - \dfrac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1-\beta}{5-\beta} & \dfrac{2}{(5-\beta)(1-\beta)} \\ -\beta \dfrac{1-\beta}{5-\beta} & -\beta \cdot \dfrac{2}{(5-\beta)(1-\beta)} - \dfrac{1}{2} \end{pmatrix}$$

$$\det(\lambda I - J) = \det \begin{pmatrix} \lambda - \dfrac{1-\beta}{5-\beta} & -\dfrac{2}{(5-\beta)(1-\beta)} \\ \beta \cdot \dfrac{1-\beta}{5-\beta} & \lambda + \beta \cdot \dfrac{2}{(5-\beta)(1-\beta)} + \dfrac{1}{2} \end{pmatrix}$$

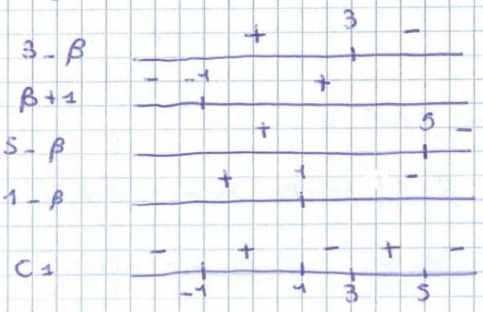
$$= \lambda^2 + \overbrace{\left\{ \frac{2\beta}{(5-\beta)(1-\beta)} + \frac{1}{2} - \frac{1-\beta}{5-\beta} \right\}}^{c_1} \lambda + \underbrace{\left(\frac{2\beta}{(5-\beta)(1-\beta)} + \frac{1}{2} \right) \frac{\beta-1}{5-\beta} + \frac{2}{(5-\beta)\cancel{(1-\beta)}} \cdot \beta \frac{\cancel{(1-\beta)}}{5-\beta}}_{c_0}$$

$$c_1 = \frac{4\beta + 5 - 6\beta + \beta^2 - (\beta-1)^2}{2(5-\beta)(1-\beta)}$$

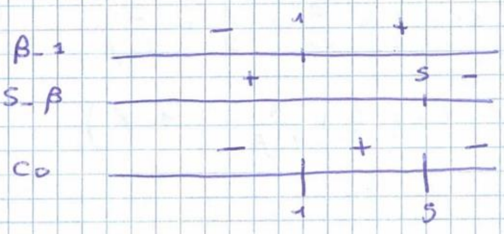
$$= \frac{4\beta + 5 - 6\beta + \beta^2 - 2\beta^2 - 2 + 4\beta}{2(5-\beta)(1-\beta)} = \frac{-\beta^2 + 2\beta + 3}{2(5-\beta)(1-\beta)} = \frac{(3-\beta)(\beta+1)}{2(5-\beta)(1-\beta)}$$

$$c_0 = -\frac{\cancel{2\beta}}{(\cancel{5-\beta})^2} + \frac{1}{2}\,\frac{\beta-1}{5-\beta} + \frac{\cancel{2\beta}}{(\cancel{5-\beta})^2} = \frac{1}{2}\cdot\frac{\beta-1}{5-\beta}$$

Segno di C_1



Segno di C_0



se $\beta \in (3,5)$, $C_0, C_1 > 0 \Rightarrow \bar{x}$ è S.d.e. Asintoticamente stabile.

se $\beta < 3$ o $\beta > 5 \Rightarrow \bar{x}$ è instabile

$\beta = 3 \Rightarrow \lambda^2 + \left(\frac{6}{-4} + \frac{1}{2} - \frac{(-2)}{2}\right)\lambda + \frac{1}{2}\cdot\frac{2}{2} = \lambda^2 + \frac{1}{2} \rightarrow \lambda = \pm j\frac{1}{\sqrt{2}}$

$\rightarrow$ non posso concludere nulla.

$\beta = 5$ NO! $\beta \neq 5$.

3) $\beta = 4$, Determinare i modi della Risp. libera del sist. linearizzato nell'intorno dei p.ti di equilibrio.

$\beta = 4$

$$\lambda^2 + \frac{(-1)5}{2\cdot 1(-3)}\lambda + \frac{1}{2}\cdot\frac{3}{1} = \lambda^2 + \frac{5}{6}\lambda + \frac{3}{2}$$

$$\lambda = \frac{-\frac{5}{6} \pm \sqrt{\frac{25}{36} - 6}}{2} = -\frac{5}{12} \pm \frac{1}{2}\sqrt{-\frac{11}{36}} = -\frac{5}{12} \pm j\frac{\sqrt{11}}{12}$$

Modi: $e^{-5/12\,t}\cos\left(\frac{\sqrt{11}}{12}t\right)$, $e^{-5/12\,t}\,\mathrm{sen}\left(\frac{\sqrt{11}}{12}t\right)$

ES. 3

$$(*) \begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) \\ \dot{x}_3(t) = -\frac{1}{2}x_1(t) - x_2(t) - x_3(t) + u(t) \\ y(t) = x_1(t) \end{cases}$$

$$u = \begin{cases} 1 & \text{se} \quad y > 1 \\ y & \text{se} \quad -1 \leq y \leq 1 \\ -1 & \text{se} \quad y < -1 \end{cases} \quad (**)$$

Determinare gli S.d.e.

$$\begin{cases} x_2 = 0 \\ x_3 = 0 \\ -\frac{1}{2}x_1 - x_2 - x_3 + u = 0 \end{cases} \to \frac{1}{2}x_1 = u \to 2u = x_1$$

u: funzione non lineare.

Accoppiamento di un sistema lineare $(*)$ con un sistema non lineare $(**)$ => Sist. non lineare.

1° caso: $y = x_1 > 1 \to u = 1 \to x_1 = 2$

$\to \bar{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ è S.d.e.

2° caso: $-1 \leq x_1 \leq 1 \to u = x_1 \to x_1 = 2x_1 \to x_1 = 0$

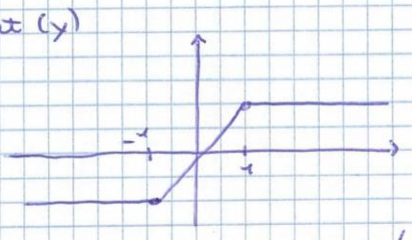
$\to \bar{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ è S.d.e.

3° caso: $y = x_1 < -1 \to u = -1 \to x_1 = -2$

$\to \bar{x} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$ è S.d.e.

Studiare la stabilità degli S.d.e.

Sat (y)



Caso 1 $\quad u = 1 \quad \bar{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

$$J = \frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & -1 & -1 \end{pmatrix}$$

$$\det(\lambda I - J) = \det\begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ \frac{1}{2} & 1 & \lambda+1 \end{pmatrix} = \lambda(\lambda^2+\lambda+1) + \frac{1}{2} = \lambda^3 + \lambda^2 + \lambda + \frac{1}{2}$$

$$\begin{array}{c|cc} 3 & 1 & 1 \\ 2 & 1 & 1/2 \\ 1 & 1/2 & \\ 0 & 1/2 & \end{array}$$

$\Rightarrow \bar{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ è S.d.e. Asintoticamente Stabile.

Caso 3 ... la matrice J è uguale!
Quindi anche $\bar{x} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$ è S.d.e. Asintoticamente Stabile.

Caso 2 $\quad u = y = x_1 \qquad \bar{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -\frac{1}{2}x_1 - x_2 - x_3 + x_1 = \frac{1}{2}x_1 - x_2 - x_3 \end{cases}$$

$$J = \frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & -1 & -1 \end{pmatrix}$$

$\det(\lambda I - J) = \lambda^3 + \lambda^2 + \lambda - \frac{1}{2}$ $\quad\leftarrow$ 1 Var. di Segno

$\rightarrow \bar{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ è INSTABILE $\quad\leftarrow$ Per Cartesio!

ES. 4

$$x_1(k+1) = x_1(k) + \delta x_2(k)$$
$$x_2(k+1) = x_2(k) + \delta u(k)$$

$$x(k+1) = \underbrace{\begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix}}_{A} x(k) + \underbrace{\begin{bmatrix} 0 \\ \delta \end{bmatrix}}_{B} u(k)$$

$$x(0) = \begin{pmatrix} k \\ 0 \end{pmatrix} \longrightarrow x(T) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \min T$$

$$x(T) - A^T x(0) = [\,B \;\; AB \;\; A^2B \;\; \dots \;\; A^{T-1}B\,] \begin{bmatrix} u(T-1) \\ u(T-2) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix}$$

$$T = 1 \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} - A \begin{pmatrix} k \\ 0 \end{pmatrix} = \begin{bmatrix} 0 \\ \delta \end{bmatrix} u(0)$$

$$-\begin{pmatrix} k \\ 0 \end{pmatrix} = \begin{bmatrix} 0 \\ \delta \end{bmatrix} u(0) \longrightarrow \text{no!}$$

$$T = 2 \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} - A^2 \begin{pmatrix} k \\ 0 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & \delta^2 \\ \delta & \delta \end{bmatrix}}_{R} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

$$-\begin{pmatrix} k \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & \delta^2 \\ \delta & \delta \end{pmatrix} \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} \quad R \rightarrow \text{Rank}\,R = 2$$

Sistema completamente Raggiungibile.

$$\begin{pmatrix} u(1) \\ u(0) \end{pmatrix} = \begin{pmatrix} 0 & \delta^2 \\ \delta & \delta \end{pmatrix}^{-1} \begin{pmatrix} -k \\ 0 \end{pmatrix} = \frac{1}{-\delta^3} \begin{pmatrix} \delta & -\delta^2 \\ -\delta & 0 \end{pmatrix} \begin{pmatrix} -k \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} k/\delta^2 \\ -k/\delta^2 \end{pmatrix} = \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}$$

$T = 10$

$$R_{10} = [\,B \;\; AB \;\; A^2B \;\; \dots \;\; A^9B\,]$$

$$= \begin{bmatrix} 0 & \delta^2 & 2\delta^2 & 3\delta^2 & \cdots & 9\delta^2 \\ \delta & \delta & \delta & \delta & \cdots & \delta \end{bmatrix}$$

$$\underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix} - A^{10} \begin{pmatrix} k \\ 0 \end{pmatrix}} = R_{10} \begin{bmatrix} u(9) \\ u(8) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix}$$

$$\begin{pmatrix} -k \\ 0 \end{pmatrix} = \begin{bmatrix} 0 & \delta^2 & \cdots & 9\delta^2 \\ \delta & \delta & \cdots & \delta \end{bmatrix} \begin{bmatrix} u(9) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix}$$

$$\begin{cases} u(0) + u(1) + \dots + u(9) = 0 \\ u(8) + 2u(7) + 3u(6) + \dots + 9u(0) = -\dfrac{k}{\delta^2} \end{cases}$$

∞^8 soluzioni

minimizzare l'energia

$$\min \sum_{i=0}^{9} u^2(i)$$

$$u^* = Q_{10}^T (Q_{10} Q_{10}^T)^{-1} \begin{pmatrix} -k \\ 0 \end{pmatrix}.$$

$$\begin{bmatrix} 0 & \delta^2 & 2\delta & \dots & 9\delta^2 \\ \delta & \delta & \delta & \dots & \delta \end{bmatrix} \begin{bmatrix} 0 & \delta \\ \delta^2 & \delta \\ 2\delta^2 & \delta \\ \vdots & \vdots \\ 9\delta^2 & \delta \end{bmatrix} = \begin{bmatrix} \delta^4(1+4+9+\dots+81) & \delta^3(1+2+\dots+9) \\ \delta^3(1+2+\dots+9) & \delta^2(10) \end{bmatrix}$$

...

ES. 1

$x(k+1) = E\bar{x}(k)$

$$\bar{x}(k) = x(k) - \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix} x(k) + \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} u(k)$$

$$x(k+1) = \underbrace{E(I-R)}_{A} x(k) + \underbrace{E \cdot S}_{B} u(k)$$

caso a) $[R = 0]$

$$A = E = \begin{pmatrix} 0,5 & 0,4 & 0,2 \\ 0,3 & 0,4 & 0,7 \\ 0,2 & 0,2 & 0,2 \end{pmatrix} \qquad B = E \cdot S = \begin{pmatrix} 0,34 \\ 0,5 \\ 0,2 \end{pmatrix}$$

Caso b) $[R = \mathrm{diag}(0,5 \;\; 0,3 \;\; 0,1)]$

$A = E(I-R) = \dots$

$u(k) = r^T x(k)$

$x(k+1) = Ax(k) + Br^T x(k) = (A + Br^T) x(k)$

Caso a) $\to A + Br^T = E$

Caso b) $\to A + Br^T = E(I-R) + E \cdot s \cdot r^T$

$y(k) = x_1(k) + x_2(k) + x_3(k) = C x(k) \qquad C = [1 \; 1 \; 1]$

$y_e(k) = C \cdot (A + Br^T)^k \cdot x(0) = \gamma_1 m_1(k) + \gamma_2 m_2(k) + \gamma_3 m_3(k)$

caso a) autov. di E $1{,}02 \quad -0{,}02 \quad 0{,}1$

caso b) autov. di $A+Br^T$ $1{,}03 \quad 0{,}04 \quad 0{,}02$

caso a) modi $(1{,}02)^k, \ (-0{,}02)^k, \ (0{,}1)^k$

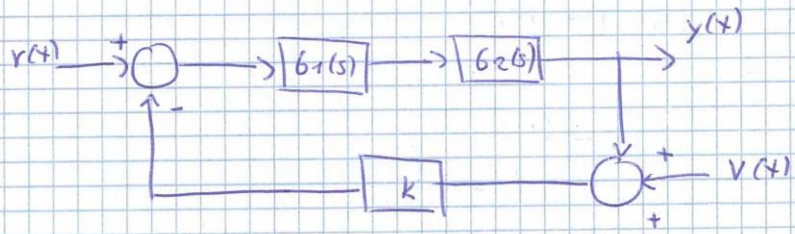
caso b) modi $(1{,}03)^k, \ (0{,}04)^k, \ (0{,}02)^k$

Considero i modi divergenti:

$(1{,}02)^k$ ← Economia cresce del 2%

$(1{,}03)^k$ ← Economia cresce del 3%

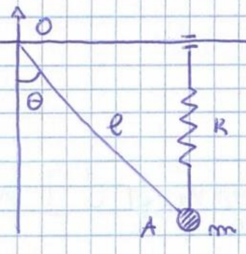
ES. 2 PROVA ITINERE



$W_1(s)$ da $v(t)$ a $y(t)$

$$= \frac{-KG_1G_2}{1+KG_1G_2}$$

ES. 1 ELABORATO 5



1) Determinare una rappr. del sistema assumendo come var. di stato $\theta(t)$, $\dot{\theta}(t)$

$$Ml\ddot{\theta}(t) = Mg\,\mathrm{sen}(\theta(t)) - 2K\cos(\theta(t))\,\mathrm{sen}(\theta(t))$$

$$\ddot{\theta}(t) = \frac{g}{l}\,\mathrm{sen}(\theta(t)) - \frac{K}{M}\cos(\theta(t))\,\mathrm{sen}(\theta(t))$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$$

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = \frac{g}{l}\,\mathrm{sen}(x_1(t)) - \frac{K}{M}\cos(x_1(t))\,\mathrm{sen}(x_1(t))$$

2) Determinare gli S. d. e.

$$\begin{cases} x_2 = 0 \\ \frac{g}{l}\,\mathrm{sen}\,x_1 - \frac{K}{M}\cos x_1\,\mathrm{sen}\,x_1 = 0 \end{cases}$$

$$\mathrm{sen}\,x_1\left(\frac{g}{l} - \frac{K}{M}\cos x_1\right) = 0$$

$$\bar{x}_1(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \bar{x}_2(t) = \begin{pmatrix} \pi \\ 0 \end{pmatrix} \qquad \bar{x}_3(t) = \begin{pmatrix} \arccos\left(\frac{gM}{lK}\right) \\ 0 \end{pmatrix}$$

3) Determinare i sistemi linearizzati

$$\Delta x = x - \bar{x}$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos x_1 - \frac{K}{M}\underbrace{(-\mathrm{sen}^2 x_1 + \cos^2 x_1)}_{\cos(2x_1)} & 0 \end{bmatrix}$$

$$\Delta x_1(t) = x(t)$$

$$\Delta x_2(t) = x(t) - \begin{pmatrix} \pi \\ 0 \end{pmatrix}$$

$$\Delta x_3(t) = x(t) - \begin{pmatrix} \arccos\left(\frac{gM}{lK}\right) \\ \end{pmatrix}$$

$$\Delta \dot{x}_1(t) = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} - \frac{k}{M} & 0 \end{bmatrix} \Delta x_1(t)$$

$$\Delta \dot{x}_2(t) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} - \frac{k}{M} & 0 \end{bmatrix} \Delta x_2(t)$$

$$\Delta \dot{x}_3(t) = \begin{bmatrix} 0 & 1 \\ \frac{g^2 M}{l^2 k} - \frac{k}{M} \cos\left(2 \arccos\left(\frac{gm}{lk}\right)\right) & 0 \end{bmatrix} \Delta x_3(t)$$

$$= \begin{bmatrix} 0 & 1 \\ \frac{g^2 M}{l^2 k} - \frac{k}{M}\left(2 \frac{g^2 m^2}{l^2 k^2} - 1\right) & 0 \end{bmatrix} \Delta x_3(t)$$

$$= \begin{bmatrix} 0 & 1 \\ -\frac{g^2 M}{l^2 k} + \frac{k}{m} & 0 \end{bmatrix} \Delta x_3(t)$$

4) Per ogni sistema linearizzato, si determino i parametri per i quali i modi risultano puramente oscillanti

Per determinare il carattere dei modi analizziamo la $x_l(t)$. $\qquad x_l(s) = (sI - A)^{-1} x(0)$

Per $\bar{x}_1$: $\quad \det[(sI - A)^{-1}] = s^2 - \frac{g}{l} + \frac{k}{M} = 0$

$s^2 = -\frac{k}{M} + \frac{g}{l} \implies s = \pm\sqrt{\frac{k}{M} + \frac{g}{l}}$ ← La radice deve essere < 0 per avere modi del tipo $e^{\alpha t}\cos(\omega t)$

$-\frac{k}{M} + \frac{g}{l} < 0 \implies k > \frac{Mg}{l}$

Per $\bar{x}_2$: $\quad \det[(sI - A)^{-1}] = s^2 + \frac{g}{l} + \frac{k}{M} = 0$

$s = \pm\sqrt{\frac{g}{l} + \frac{k}{M}} \qquad \frac{g}{l} + \frac{k}{M} < 0$

$\implies k < -\frac{g}{l} \cdot M$

Per $\bar{x}_3$: $\quad \det[(sI - A)^{-1}] = s^2 + \frac{g^2 M}{l^2 k} - \frac{k}{M} = 0$

$s = \pm\sqrt{\frac{-g^2 M}{l^2 k} + \frac{k}{M}} \qquad -\frac{g^2 M}{l^2 k} + \frac{k}{M} < 0$

$k < -\frac{Mg}{l} \vee 0 < k < \frac{Mg}{l}$

ELABORATO 5 ES. 2

b)
$$\begin{cases} \dot{x}_1(t) = -\alpha x_1(t) + x_1^2(t)\, x_2(t) + u(t) \\ \dot{x}_2(t) = x_1(t) - x_1^2(t)\, x_2(t) \end{cases} \qquad u(t) = 1 \;,\; \forall t \geq 0$$

Si calcolano gli S.d.e.

$f(\bar{x}, \bar{u}) = 0$

$$\begin{cases} -\alpha x_1(t) + x_1^2(t)\, x_2(t) + 1 = 0 \\ x_1(t) - x_1^2(t)\, x_2(t) = 0 \end{cases} \qquad \begin{cases} x_1(t) = 1/\alpha - 1 \\ x_2(t) = \alpha - 1 \end{cases}$$

$$\bar{x} = \begin{pmatrix} 1/\alpha - 1 \\ \alpha - 1 \end{pmatrix}$$

applico il criterio ridotto di Lyapunov.

$$A = \frac{\partial f}{\partial x}\Big|_{x=\bar{x}} = \begin{pmatrix} -\alpha + 2x_1(t)x_2(t) & x_1(t)^2 \\ 1 - 2x_1(t)x_2(t) & -x_1(t)^2 \end{pmatrix}\Bigg|_{x=\bar{x}}$$

$$= \begin{pmatrix} -\alpha + 2 & 1/(\alpha-1)^2 \\ 1-2 & -1/(\alpha-1)^2 \end{pmatrix}$$

$$(\lambda I - A) = \begin{pmatrix} \lambda + \alpha - 2 & -1/(\alpha-1)^2 \\ 1 & \lambda + 1/(\alpha-1)^2 \end{pmatrix}$$

$$\det(\lambda I - A) = (\lambda + \alpha - 2)\left(\lambda + \frac{1}{(\alpha-1)^2}\right) + \frac{1}{(\alpha-1)^2}$$

$$= \lambda^2 + \frac{\lambda}{(\alpha-1)^2} + (\alpha-2)\lambda + (\alpha-2)\cdot\frac{1}{(\alpha-1)^2} + \frac{1}{(\alpha-1)^2} = 0$$

$$\lambda^2(\alpha-1)^2 + \lambda + (\alpha-2)(\alpha-1)^2\lambda + (\alpha-2) + 1 = 0$$

$$\lambda^2(\alpha-1)^2 + \lambda\left((\alpha-2)(\alpha-1)^2 + 1\right) + (\alpha-1) = 0$$

Per Cartesio:

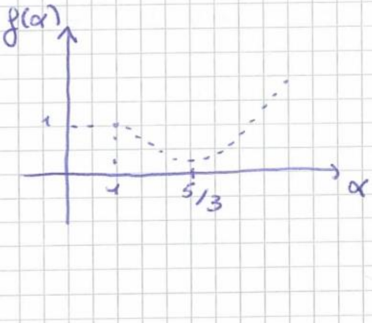
$$\begin{cases} (\alpha-1)^2 > 0 \\ (\alpha-2)(\alpha-1)^2 + 1 > 0 \\ \alpha - 1 > 0 \end{cases} \qquad \text{S.d.e. As. Stabile } \alpha > 1.$$

$\alpha^3 - 4\alpha^2 + 5\alpha - 1 > 0$

$3\alpha^2 - 8\alpha + 5 = \alpha^2 - \frac{8}{3}\alpha + \frac{5}{3}$

$= (\alpha - 1)\left(\alpha - \frac{5}{3}\right)$

$f\left(\frac{5}{3}\right) > 0$



c)

$$\begin{cases} \dot{x}_1(t) = -x_1(t) - 2x_2(t) \\ \dot{x}_2(t) = 4x_1(t) + \alpha x_2(t) \end{cases}$$

$$\begin{cases} -x_1(t) - 2x_2(t) = 0 \\ 4x_1(t) + \alpha x_2(t) = 0 \end{cases} \qquad \begin{cases} x_1(t) = -2x_2(t) = 0 \\ -8x_2(t) + \alpha x_2(t) = 0 \end{cases} \qquad \bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_2(t)(-8+\alpha) = 0$$

$$x_2(t) = 0$$

$$A = \begin{pmatrix} -1 & -2 \\ 4 & \alpha \end{pmatrix} \qquad (\lambda I - A) = \begin{pmatrix} \lambda + 1 & 2 \\ -4 & \lambda - \alpha \end{pmatrix}$$

$$\det(\lambda I - A) = (\lambda+1)(\lambda-\alpha) + 8 = \lambda^2 - \alpha\lambda + \lambda - \alpha + 8 = 0$$

$$= \lambda^2 + \lambda(1-\alpha) + (8-\alpha) = 0$$

$$\lambda_{1/2} = \frac{-(1-\alpha) \pm \sqrt{(1-\alpha)^2 - 4(8-\alpha)}}{2} = \frac{\alpha - 1 \pm \sqrt{1 + \alpha^2 - 2\alpha - 32 + 4\alpha}}{2}$$

$$= \frac{\alpha - 1 \pm \sqrt{\alpha^2 + 2\alpha - 31}}{2}$$

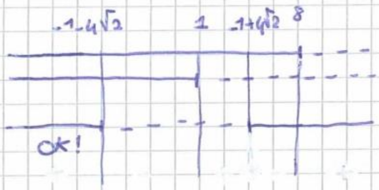
$$\alpha^2 + 2\alpha - 31 \geq 0 \implies \lambda \in \mathbb{R}$$

$$\alpha = \frac{-2 \pm \sqrt{4 + 124}}{2} = \frac{-2 \pm \sqrt{+128}}{2} = -1 \pm 4\sqrt{2}$$

$$\alpha \leq -1 - 4\sqrt{2} \quad \vee \quad \alpha \geq -1 + 4\sqrt{2}$$

$$-1 + \alpha + \sqrt{\alpha^2 + 2\alpha - 31} < 0 \implies \cancel{\alpha^2} + 2\alpha - 31 < 1 - 2\alpha + \cancel{\alpha^2} \quad , \alpha < 1$$

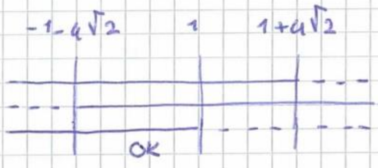
$$\alpha < 8$$



Per $\alpha \leq -1 - 4\sqrt{2}$ S.d.e. stabile. As.

$$-1 - 4\sqrt{2} < \alpha < 1 + 4\sqrt{2} \implies \lambda \in \mathbb{C}$$

$$\alpha - 1 < 0 \quad , \alpha < 1$$



Per $-1 - 4\sqrt{2} < \alpha < 1$ S.d.e. Stabile. As.

d)
$$\begin{cases} x_1(k+1) = x_2(k) \\ x_2(k+1) = -\alpha^2 x_1(k) + 2\alpha x_2(k) \end{cases}$$

Calcoliamo gli S.d.e.

$f(\bar{x}, \bar{u}) - \bar{x} = 0$

$$\begin{cases} \bar{x}_1 = \bar{x}_2 \\ \bar{x}_2 = -\alpha^2 \bar{x}_1 + 2\alpha \bar{x}_2 \end{cases} \qquad \bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\bar{x}_1 = -\alpha^2 \bar{x}_1 + 2\alpha \bar{x}_1 \Rightarrow \bar{x}_1 (1 + \alpha^2 - 2\alpha) = 0$

$\bar{x}_1 = 0$.

Verifico la stabilità degli S.d.e.

$$A = \begin{pmatrix} 0 & 1 \\ -\alpha^2 & 2\alpha \end{pmatrix} \qquad (\lambda I - A) = \begin{pmatrix} \lambda & -1 \\ \alpha^2 & \lambda - 2\alpha \end{pmatrix}$$

$\det(\lambda I - A) = \lambda(\lambda - 2\alpha) + \alpha^2(+1) = \lambda^2 - 2\alpha\lambda + \alpha^2 = (\lambda - \alpha)^2 = 0$

$\lambda_1 = \alpha$ Per $|\alpha| < 1$: S.d.e. stabile As.

$\lambda_2 = \alpha$ Per $|\alpha| > 1$: S.d.e. Instabile

e)
$$\begin{cases} x_1(k+1) = x_2(k) + u(k) \\ x_2(k+1) = x_1(k) - \alpha x_2(k) \end{cases} \qquad u(k) = 1 \;, \forall k \geq 0$$

$$\begin{cases} \bar{x}_1 = \bar{x}_2 + 1 = \frac{1}{\alpha} + 1 = \frac{\alpha+1}{\alpha} \\ \bar{x}_2 = \bar{x}_1 - \alpha \bar{x}_2 \end{cases} \qquad \bar{x} \begin{pmatrix} \frac{\alpha+1}{\alpha} \\ \frac{1}{\alpha} \end{pmatrix}$$

$\cancel{\bar{x}_2} = \cancel{\bar{x}_2} + 1 - \alpha \bar{x}_2$

$\bar{x}_2 = \frac{1}{\alpha}$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & -\alpha \end{pmatrix} \qquad (\lambda I - A) = \begin{pmatrix} \lambda & -1 \\ -1 & \lambda + \alpha \end{pmatrix}$$

$\det(\lambda I - A) = \lambda(\lambda + \alpha) - 1 = \lambda^2 + \alpha\lambda - 1$

$$\lambda_{1/2} = \frac{-\alpha \pm \sqrt{\alpha^2 + 4}}{2}$$

$\forall \alpha \Rightarrow \lambda \in \mathbb{R}$

$|\lambda_{1/2}| < 1$ S.d.e. Stabile As.

$|\lambda_{1/2}| > 1$ S.d.e. Instabile

f)

$$\begin{cases} x_1(k+1) = \dfrac{x_1^2(k)x_2(k) - x_1^2(k) - x_2(k) - 2x_1(k) + 2}{x_1(k) - 2} \\ x_2(k+1) = 2x_1(k)x_2(k) + x_2(k) \end{cases}$$

$$\begin{cases} \bar{x}_1 = \dfrac{\bar{x}_1^2\bar{x}_2 - \bar{x}_1^2 - \bar{x}_2 - 2\bar{x}_1 + 2}{\bar{x}_1 - 2} \\ \cancel{\bar{x}_2} = 2\bar{x}_1\bar{x}_2 + \cancel{\bar{x}_2} \end{cases}$$

$$\bar{x}_1(\bar{x}_1 - 2) = \bar{x}_1^2\bar{x}_2 - \bar{x}_1^2 - \bar{x}_2 - 2\bar{x}_1 + 2$$

$$\bar{x}_1^2 - \cancel{2\bar{x}_1} = \bar{x}_1^2\bar{x}_2 - \bar{x}_1^2 - \bar{x}_2 - \cancel{2\bar{x}_1} + 2$$

$$\bar{x}_1^2 - \bar{x}_1^2\bar{x}_2 + \bar{x}_1^2 + \bar{x}_2 - 2 = 0$$

$$2\bar{x}_1^2 - \bar{x}_1^2\bar{x}_2 + \bar{x}_2 - 2 = 0$$

$$2\bar{x}_1\bar{x}_2 = 0$$

$$\begin{cases} \bar{x}_1 = 0 \\ \bar{x}_2 = 2 \end{cases} \qquad \begin{cases} \bar{x}_2 = 0 \\ \cancel{2}\bar{x}_1^2 = \cancel{2} \end{cases} \qquad \begin{cases} \bar{x}_2 = 0 \\ \bar{x}_1 = -1 \end{cases} \qquad \begin{cases} \bar{x}_2 = 0 \\ \bar{x}_1 = 1 \end{cases}$$

$$\bar{x}_A = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \qquad \bar{x}_B = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad \bar{x}_C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A = \left.\begin{pmatrix} \dfrac{x_1^2x_2 - 3x_1^2 - 4x_1x_2 - 2x_1 - x_2 + 6}{(x_1-2)^2} & \dfrac{x_1^2}{x_1-2} - \dfrac{1}{x_1-2} \\ 2x_2 & 2x_1+1 \end{pmatrix}\right|_{x=\bar{x}}$$

$$A\Big|_{x=\bar{x}_A} = \begin{pmatrix} \frac{4}{4} = 1 & 1/2 \\ 4 & 1 \end{pmatrix} \qquad \lambda I - A = \begin{pmatrix} \lambda - 1 & -1/2 \\ -4 & \lambda - 1 \end{pmatrix}$$

$$\det(\lambda I - A) = (\lambda - 1)^2 - 2 = 0$$

$$= \lambda^2 + 1 - 2\lambda - 2 = 0$$

$$\lambda^2 - 2\lambda - 1 = 0 \qquad \lambda_{1/2} = 1 \pm \sqrt{2}$$

$\bar{x}_A$ S.d.e. Instabile

$$A\Big|_{x=\bar{x}_C} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \qquad \lambda I - A = \begin{pmatrix} \lambda - 1 & 0 \\ 0 & \lambda - 3 \end{pmatrix}$$

$$\det(\lambda I - A) = (\lambda - 1)(\lambda - 3) = 0$$

$$\begin{cases} \lambda = 1 \\ \lambda = 3 \end{cases}$$

$\bar{x}_C$ S.d.e Instabile

$$A\Big|_{x=\bar{x}_3} = \begin{pmatrix} 5/9 & 0 \\ 0 & -1 \end{pmatrix} \qquad \lambda I - A = \begin{pmatrix} \lambda - 5/9 & 0 \\ 0 & \lambda + 1 \end{pmatrix}$$

$$\det(\lambda I - A) = (\lambda - 5/9)(\lambda + 1) = 0$$

$$\begin{cases} \lambda_1 = 5/9 \\ \lambda_2 = -1 \end{cases}$$

$|\lambda_1| < 1$ e $|\lambda_2| \leq 1$, dato che molt. alg (λ_2) = molt. geom (λ_2) $= 1$

$\Rightarrow$ S. d. e. $\bar{x}_3$?

a)

$$\dot{x}(t) = \cos x(t) + \frac{2}{\pi} x(t) + u(t) \qquad \text{per } u(t) = -1, \ \forall t \geq 0.$$

Calcoliamo gli S. d. e :

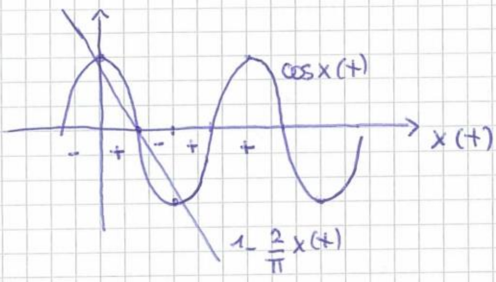
$$\cos x(t) + \frac{2}{\pi} x(t) - 1 = 0$$

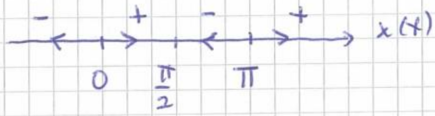
$x(t) = 0$

$x(t) = \frac{\pi}{2}$

$x(t) = \pi$

$$\cos x(t) - \left(1 - \frac{2}{\pi} x(t)\right) > 0$$





$x(t) = 0$ S. d. e instabile

$x(t) = \pi/2$ S. d. e. stabile

$x(t) = \pi$ S. d. e instabile.