SISTEMI DINAMICI

[Formulario Esame]

A CURA DI ALESSANDRO PAGHI

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LINK AL CORSO:
http://www3.diism.unisi.it/FAC/index.php?bodyinc=didattica/inc.insegnamento.php&id=54893&aa=2014

FREQUENTAZIONE: Consigliata.

TRASFORMATA DI LAPLACE

$$F(s) = \mathcal{L}[f(t)] = \int_0^{+\infty} f(t)\, e^{-st}\, dt$$

$f(t) = \alpha_1 f_1(t) + \alpha_2 f_2(t) \Rightarrow F(s) = \alpha_1 F_1(s) + \alpha_2 F_2(s)$

$g(t) = f(t-\Delta) \cdot 1(t-\Delta) \Rightarrow G(s) = F(s)\, e^{-\Delta s}$

$g(t) = f(t)\, e^{\gamma t} \Rightarrow G(s) = F(s-\gamma)$

$f(t) = 1(t) \Rightarrow F(s) = \frac{1}{s}$

$f(t) = \delta(t) \Rightarrow F(s) = 1$

$g(t) = e^{\alpha t} \Rightarrow G(s) = \frac{1}{s-\alpha}$

$g(t) = t \cdot f(t) \Rightarrow G(s) = -\frac{d}{ds} F(s)$

$f(t) = r_k(t) = \frac{t^k}{k!} \cdot 1(t) \Rightarrow F(s) = \frac{1}{s^{k+1}}$

$f_k(t) = \frac{t^k}{k!} e^{\alpha t} \Rightarrow F_k(s) = \frac{1}{(s-\alpha)^{k+1}}$

$f(t) = \cos(\omega t) \Rightarrow F(s) = \frac{s}{s^2+\omega^2}$

$f(t) = \mathrm{sen}(\omega t) \Rightarrow F(s) = \frac{\omega}{s^2+\omega^2}$

$g(t) = f'(t) \cdot 1(t) \Rightarrow G(s) = s \cdot F(s) - f(0)$

$g(t) = \int_0^t f(\tau)\, d\tau \Rightarrow G(s) = \frac{1}{s} \cdot F(s)$

CASI FREQUENTI:

$f(t) = t \cdot 1(t) \Rightarrow F(s) = \frac{1}{s^2}$

$f(t) = t^n \cdot 1(t) \Rightarrow F(s) = (-1)^n \frac{d^n}{ds^n} F(s)$

$f(t) = t\, e^{-t} \Rightarrow F(s) = \frac{1}{(s+1)^2}$

$f(t) = e^{-t} \cos(t) \Rightarrow F(s) = \frac{s+1}{(s+1)^2+1}$

$$\left.\begin{aligned} f(t) &= e^{-\frac{t}{2}} \cos\left(\tfrac{\sqrt{3}}{2} t\right) \Rightarrow F(s) = \frac{s+1/2}{(s+1/2)^2 + 3/4} \\ f(t) &= e^{-\frac{t}{2}}\, \mathrm{sen}\left(\tfrac{\sqrt{3}}{2} t\right) \Rightarrow F(s) = \frac{\sqrt{3}/2}{(s+1/2)^2 + 3/4} \end{aligned}\right\} \begin{aligned} &\text{Da } s^2+s+1=0 \\ &p_{1/2} = -\tfrac{1}{2} \pm j\tfrac{\sqrt{3}}{2} \end{aligned}$$

$P(t) = f(t) * g(t) \Rightarrow P(s) = F(s) \cdot G(s)$

TRASFORMATA Z

$$F(z) = \mathcal{Z}[f(k)] = \sum_{k=0}^{+\infty} f(k) z^{-k} = f(0) + f(1) z^{-1} + f(2) z^{-2} + \dots$$

$f(k) = \delta(k) \quad \Rightarrow F(z) = 1$

$f(k) = 1(k) \quad \Rightarrow F(z) = \frac{z}{z-1}$

$f(k) = \alpha u_1 + \beta u_2 \quad \Rightarrow F(z) = \alpha \cdot \mathcal{Z}[u_1] + \beta \cdot \mathcal{Z}[u_2]$

$g(k) = f(k-k_0) \cdot 1(k-k_0) \quad \Rightarrow G(z) = z^{-k_0} \cdot F(z)$

$g(k) = \rho^k \cdot f(k) \quad \Rightarrow G(z) = F\left(\frac{z}{\rho}\right)$

$g(k) = k \cdot f(k) \quad \Rightarrow G(z) = -z \cdot \frac{d}{dz} F(z)$

$f(k) = r_m(k) = \binom{k}{m} 1(k) \quad \Rightarrow F(z) = \frac{z}{(z-1)^{m+1}}$

$f(k) = \cos(\omega k) \quad \Rightarrow F(z) = \frac{z^2 - \cos(\omega) \cdot z}{z^2 - 2\cos(\omega) \cdot z + 1}$

$f(k) = \text{sen}(\omega k) \quad \Rightarrow F(z) = \frac{\text{sen}(\omega) \cdot z}{z^2 \cdot 2\cos(\omega) \cdot z + 1}$

$f(k) = \rho^k \binom{k}{m} \cdot 1(k) \quad \Rightarrow F(z) = \frac{\rho^m \cdot z}{(z-\rho)^{m+1}}$

$P(k) = f(k) * g(k) \quad \Rightarrow P(z) = F(z) \cdot G(z)$

CASI FREQUENTI:

$f(k) = \rho^k \cdot 1(k) \quad \Rightarrow F(z) = \frac{z}{z-\rho}$

$f(k) = (-1)^k \quad \Rightarrow F(z) = \frac{z}{z+1}$

$f(k) = \left(\frac{1}{2}\right)^k \quad \Rightarrow F(z) = \frac{z}{z - 1/2}$

$f(k) = \delta(k-1) \quad \Rightarrow F(z) = \frac{1}{z}$

$f(k) = \delta(k-2) \quad \Rightarrow F(z) = \frac{1}{z^2}$

$f(k) = \left(\frac{1}{2}\right)^{k-1} \quad \Rightarrow F(z) = \frac{1}{z - 1/2}$

$f(k) = k \cdot 1(k) \quad \Rightarrow F(z) = \frac{z}{(z-1)^2}$

$f(k) = 1(k-1) \quad \Rightarrow F(z) = \frac{1}{z-1}$

$f(k) = 1(k-2) \quad \Rightarrow F(z) = \frac{1}{z(z-1)}$

TEMPO CONTINUO

TEOREMA DEL VALORE FINALE (finito sse $Re[p_i]<0$ eccetto $p_0=0$, $\mu=1$)

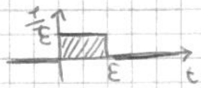
$$\lim_{t\to+\infty} f(t) \ \exists \text{ finito} \Rightarrow \lim_{t\to+\infty} f(t) = \lim_{s\to 0} s \cdot F(s)$$

GRADINO UNITARIO

$$1(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t<0 \end{cases}$$

DELTA DI DIRAC (δ)

$$\delta_\varepsilon(t) = \begin{cases} 1/\varepsilon & 0 \leq t \leq \varepsilon \\ 0 & \text{altrimenti} \end{cases}$$



PRODOTTO DI CONVOLUZIONE

$$P(t) = f(t) * g(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

REGOLA CARTESIO: POLI CONVERGENTI

$$1s^2 + (3-k)s + k$$

$1>0$

$\Rightarrow 3-k>0$

$k>0$

$0<k<3$ ← Poli convergenti

TRASFORMATE RAPPR. i/s/o

$$X(s) = X_e(s) + X_f(s)$$

$$X_e(s) = (sI - A)^{-1} x(0) \quad ; \quad X_f(s) = (sI-A)^{-1} B \cdot U(s)$$

$$Y(s) = Y_e(s) + Y_f(s)$$

$$Y_e(s) = C(sI-A)^{-1}x(0) \quad , \quad Y_f(s) = G(s) \cdot U(s)$$

$$G(s) = C(sI-A)^{-1}B + D$$

$$(sI-A)^{-1} = \frac{[Agg(sI-A)]^T}{\det(sI-A)}$$

$$Mat[2\times2] = \begin{bmatrix} z-a & b \\ c & z-d \end{bmatrix} \Rightarrow (Mat[2\times2])^{-1} = \frac{\begin{bmatrix} z-d & -b \\ -c & z-a \end{bmatrix}}{\det(Mat[2\times2])}$$

DECOMPOSIZIONE IN FRATTI SEMPLICI

$$F(s) = \frac{R_1}{s-p_1} + \frac{R_2}{s-p_2} + \dots + \frac{R_m}{s-p_m}$$

RESIDUI: $R_i = \lim_{s\to p_i} (s-p_i) F(s)$

$$\mathcal{L}^{-1}[F(s)] = f(t) = R_1 e^{p_1 t} + R_2 e^{p_2 t} + \dots + R_m e^{p_m t}$$

ANTITRASFORMATA F(s) PROPRIA

$$F(s) = \frac{s^2+5s+3}{2s^2+6s+4} = \frac{\frac{1}{2}(s^2+3s+2)+s+\frac{1}{2}}{s^2+3s+2} = \frac{1}{2} + \frac{s+1/2}{s^2+3s+2}$$

ANALISI MODALE

POLI	MODI
$p\in\mathbb{R}$, $\mu=1$	e^{pt}
$p=\sigma\pm j\omega$, $\mu=1$	$e^{\sigma t}\cos(\omega t)$, $e^{\sigma t}\sin(\omega t)$
$p\in\mathbb{R}$, $\mu>1$	$e^{pt}, te^{pt}, t^2e^{pt}, \dots, t^{\mu-1}e^{pt}$
$p=\sigma\pm j\omega$, $\mu>1$	$e^{\sigma t}\cos(\omega t)$, $e^{\sigma t}\sin(\omega t)$, $te^{\sigma t}\cos(\omega t)$, $te^{\sigma t}\sin(\omega t)$, ..., $t^{\mu-1}e^{\sigma t}\cos(\omega t)$, $t^{\mu-1}e^{\sigma t}\sin(\omega t)$

RISP. IMPULSIVA

$u(t) = \delta(t)$

$Y_f(s) = G(s)$

RISP. AL GRADINO

$u(t) = 1(t)$

$Y_f(s) = G(s)/s$

MODULO E FASE

$c = |c|e^{j\measuredangle c}$

$c = \sigma + j\omega$

$|c| = \sqrt{\sigma^2+\omega^2}$

$\measuredangle c = \arctan\left(\frac{\omega}{\sigma}\right)$

CARATTERE DI CONVERGENZA DEI MODI

→ CONVERGENTE: $Re[p_i]<0$

→ LIMITATO NON CONV.: $Re[p_i]=0$, $\mu=1$

→ DIVERGENTE: altri casi

T. DELLA RISP. IN FREQUENZA

$y_f(t) = y_T(t) + y_{PERM}(t)$

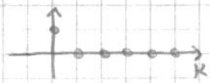
$y_p(t) = M|G(j\omega)|\cos(\omega t + \varphi + \measuredangle G(j\omega))$

$u(t) = M\cos(\omega t+\varphi)$

TEMPO DISCRETO

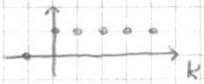
IMPULSO UNITARIO

$$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$



GRADINO UNITARIO

$$1\!\!1(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$$



PRODOTTO DI CONVOLUZIONE

$$p(k) = f(k) * g(k) = \sum_{h=0}^{k} f(h)\, g(k-h)$$

TEOREMA DEL VALORE FINALE (finito sse $|p_i| < 1$ eccetto $p_0 = 1$, $\mu = 1$)

$$\lim_{k \to +\infty} f(k)\ \exists \text{ finito} \Rightarrow \lim_{k \to +\infty} f(k) = \lim_{z \to 1} (z-1) F(z)$$

TRASFORMATE RAPPR. I/S/O

$$X(z) = X_\ell(z) + X_f(z)$$

$$X_\ell(z) = (zI - A)^{-1} z\, x(0) \; ; \; X_f(z) = (zI - A)^{-1} B \cdot U(z)$$

$$Y(z) = Y_\ell(z) + Y_f(z)$$

$$Y_\ell(z) = C (zI - A)^{-1} z\, x(0) \; ; \; Y_f(z) = G(z) \cdot U(z)$$

$$G(z) = C(zI - A)^{-1} B + D$$

RISP. IMPULSIVA

$u(k) = \delta(k)$

$Y_f(z) = G(z)$

RISP. AL GRADINO

$u(k) = 1\!\!1(k)$

$Y_f(z) = G(z) \cdot \dfrac{z}{z-1}$

ANTITRASFORMATA F(z) PROPRIA

$$F(z) = \frac{\bar{F}(z)}{z} = \sum_{i=1}^{n} \frac{R_i}{z - p_i}$$

T. RISP. IN FREQUENZA

$$y_p(t) = k |G(e^{j\omega})| \cos(\omega k + \varphi + \angle G(e^{j\omega}))$$

ANALISI MODALE

POLI	MODI
$p \in \mathbb{R}$, $\mu = 1$	$p^k \cdot 1\!\!1(k)$
$p = \rho e^{\pm j\phi}$, $\mu = 1$	$\rho^k \cos(\phi k)$, $\rho^k \sin(\phi k)$
$p \in \mathbb{R}$, $\mu > 1$	$p^k 1\!\!1(k)$, $k \cdot p^k 1\!\!1(k)$, ..., $k^{\mu-1} p^k \cdot 1\!\!1(k)$
$p = \rho e^{\pm j\phi}$, $\mu > 1$	$\rho^k \cos(\phi k)$, $\rho^k \sin(\phi k)$, $k\rho^k \cos(\phi k)$, $k\rho^k \sin(\phi k)$, ..., $k^{\mu-1}\rho^k \cos(\phi k)$, $k^{\mu-1}\rho^k \cos(\phi k)$

POLI $G(s)$, $Y_\ell(s)$

Sono gli stessi.

CARATTERE DI CONVERGENZA DEI MODI

→ CONVERGENTE: $|p_i| < 1$

→ LIMITATO NON CONV: $|p_i| = 1$, $\mu_i = 1$

→ DIVERGENTE: altri casi

INTERCONNESSIONE DI SISTEMI DINAMICI

SERIE: $G(s) = G_1(s) \cdot G_2(s)$

PARALLELO: $G(s) = G_1(s) + G_2(s)$

RETROAZIONE +: $\dfrac{G_1(s)}{1 - G_1(s) G_2(s)}$

RETROAZIONE −: $\dfrac{G_1(s)}{1 + G_1(s) G_2(s)}$

- **MASSE**
  $M \ddot{p}(t) = f(t)$
- **MOLLA**
  $f(t) = k(p_2 - p_1)$, $p_2 > p_1$
- **SMORZATORE**
  $f(t) = B(\dot{p}_2 - \dot{p}_1)$, $p_2 > p_1$
- **MASSE ROTAZIONE**
  $J \cdot \ddot{\theta}(t) = c(t)$
- **MOLLA ROTAZIONE**
  $c(t) = k(\theta_2(t) - \theta_1(t))$
- **ATTRITO DI ROTAZIONE**
  $c(t) = B(\dot{\theta}_2(t) - \dot{\theta}_1(t))$
- **CONSERVAZIONE VOLUME**
  $\dfrac{d}{dt}\{A \cdot h(t)\} = Q_i(t) - Q_u(t)$
- **CONSERVAZIONE ENERGIA**
  $v_B^2(t) = 2g\,h(t)$
- $C_f \dfrac{dT(t)}{dt} = P(t) - k_f (T(t) - T_p(t))$
- $C_p \dfrac{dT(t)}{dt} = k_f (T(t) - T_p(t)) - k_p (T_p(t) - T_e(t))$